European Journal of Finance and Banking Research Vol. 1. No. 1. 2007.

RISK AND INVESTMENT OPPORTUNITIES IN PORTFOLIO OPTIMIZATION

Ali Argun Karacabey[1]
Ankara University, Turkey.

ABSTRACT

Markowitz legendary work about portfolio optimization is accepted to be the pioneer of the modern portfolio theory. Contrary to its theoretical reputation, it has not been used extensively. Konno and Yamazaki (1991) proposed a new portfolio optimization model as an alternative to Markowitz's mean-variance model. Markowitz's mean variance model and the mean absolute deviation models regard risk in terms of deviations that may be either positive (upward) or negative (downward) in relation to the expected return. In other words both of the models penalize not only the negative (downward) deviations but also the positive (upward) deviations. In this paper, a new model that takes into consideration both risk and a better investment opportunity is proposed. The difference between the proposed model and the other portfolio optimization models is their objectives. The proposed model assumes that an investor wants to choose a portfolio with higher upside deviations and lower downside deviations.

Key word: portfolio optimization, MAD, downside risk
JEL Codes: G 11, G12, G32

[1] Faculty of Political Sciences, 06590, Ankara, Turkey.

I. INTRODUCTION

Markowitz's legendary study of portfolio optimization is regarded as the pioneering work of modern portfolio theory. In the Markowitz model, risk is stated in terms of the predicted variance of portfolio return, a function that is quadratic in the decision variables. All other functions and constraints are assumed to be linear (Sharpe, 1971). The objective of the model is to form the efficient portfolios.

Contrary to its theoretical reputation, it has not been used extensively. The two important reasons why Markowitz's model has not been implemented can be summarized as follows (Elton, Gruber, Padberg, 1976; Konno and Yamazaki, 1991): (i) the difficulty in estimating the correlation matrices, (ii) the computational difficulty of the quadratic programming model.

Sharpe (1971) claimed that if the essence of a portfolio analysis problem could be adequately captured in a form suitable for linear programming methods, the prospect for practical application would be greatly enhanced. Sharpe (1971) and Stone (1973) tried to convert the portfolio problem into a linear programming model. Konno and Yamazaki (1991) proposed a new portfolio optimization model as an alternative to Markowitz's mean-variance model. They employed L_1 – mean absolute deviation as a risk measure instead of variance, in order to overcome the problem of computational difficulty.

The MAD model is said to be a viable alternative because (i) it does not require the covariance matrix of the returns, and (ii) MAD portfolios have fewer assets (Simaan, 1997). It is also argued that as the number of assets decreases, the transaction costs of the portfolio will also decrease.

The MAD portfolio optimization model has 2T+2 rows where T is the number of periods. Feinstein and Thapa (1993) reformulated the MAD portfolio optimization model so that the number of rows decreased to T+2, which implies that the maximum number of stocks invested in decreases from 2T+2 to T+2. Chang (2005) modified Feinstein and Thapa's model so that his model has fewer variables and the same number of constraints.

All of these optimization models consider risk as the deviation of returns from the expected or mean return. However, these models assume that there is no difference between positive and negative deviation. On the other hand, for an investor positive deviation is desirable, while negative deviation is not. So, in this paper a new model that differentiates positive and negative deviations is proposed. According to this proposed model an investor simultaneously wishes to maximize positive deviations and minimize negative deviations.

II. REVIEW OF THE MEAN VARIANCE AND MAD PORTFOLIO OPTIMIZATION MODELS

Markowitz (1952) considers two rules while formulating the portfolio optimization model. First, the investor does (or should) maximize expected returns, and secondly, the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing. The concept of an efficient portfolio has emerged in accordance with these two rules.

The Markowitz portfolio optimization model employs variance as the measure of risk, and the objective of the model is to find out the weightings of assets that minimizes the variance of a portfolio and ensure a return equal to or bigger than the expected return. Accordingly the mathematical model for *n* assets is as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\
& \sum_{j=1}^{n} x_j = M_0 \\
& 0 \leq x_j \leq u_j \qquad j = 1.....n
\end{aligned}
\tag{1}
$$

Where:

σ_{ij} = covariance between assets i and j,

x_j = the amount invested in asset j,

r_j = the expected return of asset j per period,

ρ = a parameter representing the minimal rate of return required by an investor,
M_0 = total amount of the fund, and
u_j = maximum amount of money which can be invested in asset j.

Konno and Yamazaki (1991) introduced the L_1 risk function (mean absolute deviation-MAD) $w(x) = E\left|\sum R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|$ instead of the L_2 risk (variance) function where R_j is a random variable representing the rate of return per period of the asset j. They proved that these two measures are the same if $(R_1 \ldots R_n)$ are multivariate normally distributed. So the Konno-Yamazaki MAD portfolio optimization model becomes as follows:

$$\begin{aligned} \text{Minimize} \quad & w(x) = E\left|\sum R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right| \\ \text{subject to} \quad & \sum_{j=1}^{n} E[R_j] x_j \geq \rho M_0 \\ & \sum_{j=1}^{n} x_j = M_0 \\ & 0 \leq x_j \leq u_j \qquad j = 1.....n \end{aligned} \tag{2}$$

Konno and Yamazaki assumed that the expected value of the random variable can be approximated by the average from the data. So:

$$r_j = E[R_j] = \frac{1}{T}\sum_{t=1}^{T} r_{jt}$$

Where r_{jt} is the realization of random variable R_j during period t (where t=1…T). Thus, w(x) is approximated by $\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt} - r_j)x_j\right|$. Denoting $a_{jt}=r_{jt}-r_j$ (j=1…n and t=1…T), model (2) can be expressed as follows.

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n} a_{jt}x_j\right| \\ \text{subject to} \quad & \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\ & \sum_{j=1}^{n} x_j = M_0 \\ & 0 \leq x_j \leq u_j \qquad j = 1.....n \end{aligned} \tag{3}$$

Konno and Yamazaki replaced the model with model (4) which is equivalent to model (3).

$$\begin{aligned} \textit{Minimize} \quad & \frac{\sum_{t=1}^{T} y_t}{T} \\ \textit{subject to} \quad & y_t + \sum_{j=1}^{n} a_{jt}x_j \geq 0 \qquad t = 1...T \\ & y_t - \sum_{j=1}^{n} a_{jt}x_j \geq 0 \qquad t = 1...T \\ & \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\ & \sum_{j=1}^{n} x_j = M_0 \\ & 0 \leq x_j \leq u_j \qquad j = 1.....n \end{aligned} \tag{4}$$

According to Konno and Yamazaki the MAD portfolio optimization model's advantages over the Markowitz's model are (i) this model does not use the covariance matrix which therefore does not need to be calculated, (ii) solving this linear model is much easier than solving a quadratic model, (iii) the maximum number of assets that are invested in is 2T+2 (if $u_j=\infty$) while Markowitz's model may contain as many as n assets, and (iv) T can be used as a control variable to restrict the number of assets.

Feinstein and Thapa (1993) modified the MAD portfolio optimization model and proposed a new model that is equivalent to Konno and Yamazaki's but has a limit of T+2 on the number of non-zero assets in the optimal portfolio on the assumption that there is no upper limit on the investment $-u_j = \infty$ - in an asset.

They subtract non-negative surplus variables $2v_t$ and $2w_t$ from each of the constraints in problem 4 in order to replace the inequalities with equalities:

$$y_t + \sum_{j=1}^{n} a_{jt} x_j - 2v_t = 0 \tag{5}$$

$$y_t - \sum_{j=1}^{n} a_{jt} x_j - 2w_t = 0 \tag{6}$$

In order to eliminate y_t, they subtract (6) from (5) and divide by 2. Thus the portfolio optimization model becomes:

$$\begin{aligned}
&\textit{Minimize} && \sum_{t=1}^{T} (v_t + w_t) \\
&\textit{subject to} && v_t - w_t - \sum_{j=1}^{n} a_{jt} x_j = 0 \qquad t = 1 \ldots T \\
& && \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\
& && \sum_{j=1}^{n} x_j = M_0 \\
& && 0 \leq x_j \leq u_j \qquad j = 1 \ldots\ldots n \\
& && v_t \geq 0, \quad w_t \geq 0, \qquad t = 1 \ldots T
\end{aligned} \tag{7}$$

Compared with Konno and Yamazaki's optimization model under the $u_j = \infty$ assumption, model (7) allows investment in at most T+2 assets in the optimal portfolio. According to Konno and Yamazaki this means that the optimal portfolio produced by (7) should have lower transaction costs compared with the optimal portfolio obtained by model (4).

Chang (2005) reformulate Feinstein and Thapa's model by introducing a continuous variable d_t;

$$
\begin{aligned}
&\textit{Minimize} && \sum_{t=1}^{T}\left((2d_t - \sum_{j=1}^{n} a_{jt} x_j) \right) \\
&\textit{subject to} && d_t - \sum_{j=1}^{n} a_{jt} x_j \geq 0 \qquad t = 1...T \\
& && d_t \geq 0 \qquad t = 1...T \\
& && \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\
& && \sum_{j=1}^{n} x_j = M_0 \\
& && 0 \leq x_j \leq u_j \qquad j = 1.....n
\end{aligned} \tag{8}
$$

Chang (2005) proved that model (8) is equivalent to model (7) while the numbers of additional continuous variables and auxiliary sign constraints are half of the Feinstein and Thapa's model. So the CPU time and the number of iterations needed to find the optimal solution are decreased.

III. PROPOSED OPTIMIZATION MODEL

Markowitz's mean variance model and the mean absolute deviation models regard risk in terms of deviations that may be either positive (upward) or negative (downward) in relation to the expected return. In other words both of the models penalize not only the negative (downward) deviations but also the positive (upward) deviations. However, the difference between negative and positive deviations is crucial. Negative deviation is regarded as undesirable for most investors while, positive deviation is desirable.

Following the work of Markowitz (1959), in which he proposed that semivariance replace variance as the measure of risk, downside risk has been the subject of numerous studies [Grootveld and Hallebach (1999), Michalowski and Ogryczak (2001)]. The general conclusion concerning downside risk is that the left-hand side of a return distribution involves risk while the right-hand side contains the better investment opportunities Grootveld and Hallebach (1999).

Some of these studies proposed portfolio optimization models which employ downside risk as the measure of risk. A portfolio optimization model which incorporates downside risk as the measure of risk only penalizes downside deviations but does not take upside deviations into consideration. These models are similar to the mean variance or MAD models as all of them share the objective of minimizing risk, the left-hand side of a return distribution. Perhaps the most important deficiency of these models is that they do not take into consideration the *better investment opportunities*, the right hand side of a return distribution.

In this paper a new model that takes into consideration both risk and better investment opportunities are proposed. The difference between the proposed model and the other portfolio optimization models is their objectives. The proposed model assumes that an investor wants to choose a portfolio with higher upside deviations and lower downside deviations. In other words he/she has two simultaneous objectives. First he/she wants to maximize upside deviations and second wants to minimize downside deviations. These two objectives can be merged and restated as the single objective of maximizing the difference between the upside and downside deviations.

For n securities (j=1…n) during T periods (t=1…T) the downside risk (NMAD) and better investment opportunities (PMAD) can be shown in this way:

$$NMAD_{jt} = N_{jt} = \left| \min [0, r_{jt}-r_j] \right| \qquad (9)$$

$$PMAD_{jt} = P_{jt} = \left| \max [0, r_{jt}-r_j] \right| \qquad (10)$$

According to these definitions, the objective of the proposed model can be stated as follows:

$$\begin{aligned} &\textit{Maximize} \quad \sum_{t=1}^{T}\sum_{j=1}^{n} P_{jt}.x_j - \sum_{t=1}^{T}\sum_{j=1}^{n} N_{jt}.x_j \\ &\textit{subject to} \quad \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\ &\qquad \sum_{j=1}^{n} x_j = M_0 \qquad (11) \\ &\qquad 0 \leq x_j \leq u_j \qquad j = 1.....n \end{aligned}$$

This model is similar to model (12) if we assume that $\sum_{j=1}^{n} P_{jt}.x_j / T = Y_j$ and $\sum_{j=1}^{n} N_{jt}.x_j / T = Z_j$.

$$
\begin{aligned}
& \textit{Maximize} \sum_{t=1}^{T} Y_j - \sum_{t=1}^{T} Z_j \\
& \textit{subject to} - \sum_{j=1}^{n} P_{jt} x_{jt} - Y_t = 0 \qquad t = 1 \ldots T \\
& \sum_{j=1}^{n} N_{jt} x_{jt} - Z_t = 0 \qquad t = 1 \ldots T \\
& \sum_{j=1}^{n} r_j x_j \geq \rho M_0 \\
& \sum_{j=1}^{n} x_j = M_0 \\
& 0 \leq x_j \leq u_j \qquad j = 1 \ldots\ldots n
\end{aligned}
\tag{12}
$$

This new model has 2T+n variables and 2T+2 constraints. As a consequence of developments in computer technology, the number of variables and constraints are no longer a crucial problem. But this new model differs from other portfolio optimization models since it considers risk in terms of negative deviations from the mean and also takes the better investment opportunities into consideration. According to other optimization models, the security with the lowest deviation among securities with the same return is regarded as the most desirable whereas this model regards a security which has higher positive and lower negative deviations as more desirable. In other words, this new model tries to find an equilibrium point between risk and better investment opportunities.

III. COMPARISON OF THE OPTIMIZATION MODELS

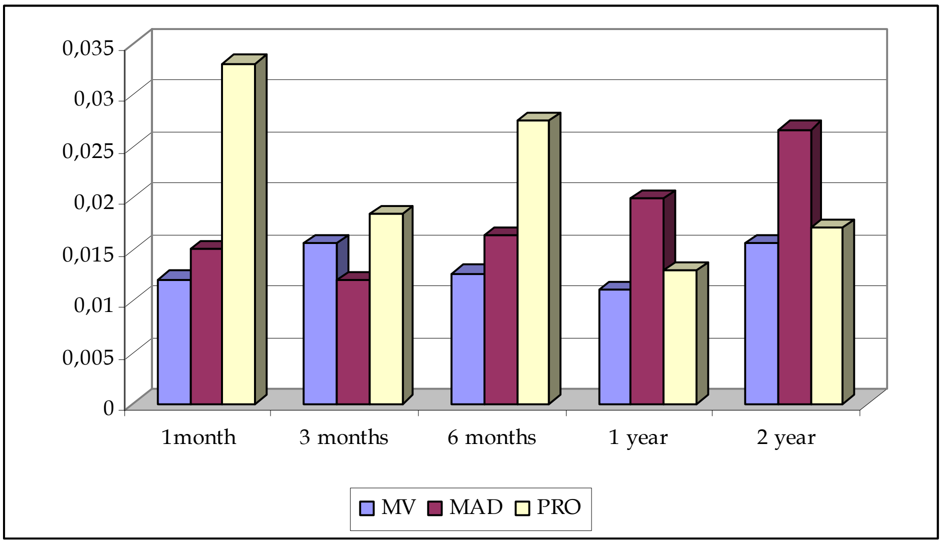
In this section, in order to evaluate the performance of the portfolio optimization models, portfolios developed according to the proposed model, mean variance and mean absolute deviation models are compared for different investment horizons. The database consists of stocks included in the ISE-100, the well-known index of the Istanbul Stock Exchange on the starting date of the analysis, January 2000. 9 of the stocks are excluded because of missing data.

In the first stage of the analysis, 48 different data sets – A1,A2…A48- which include data for 91 stocks over 12 months are prepared. A1 includes data from January 2000 to December 2000 (1-12 months); A2 includes data from February 2000 to January 2001 (2-13 months) and so on. For every data set, the mean-variance, MAD and proposed optimal portfolios are put together using models 1, 7 and 12.

In the second stage, in order to take into consideration the effect of the portfolio horizon, the assumption involving modification of the portfolios every month is replaced with modification of the portfolios in every 3 months, 6 months, 1 year and 2 years. For each assumption, the returns of the 3 portfolios developed according to the models 1, 7 and 12, are calculated, and then these returns are compared.

Portfolios modified more than once a year show that the proposed model's returns definitely exceed the returns of the other two models (Figure 1). When the investment horizon exceeds 1 year, the mean absolute deviation models produce higher returns.

Figure 1: Returns of the models for different investment horizons



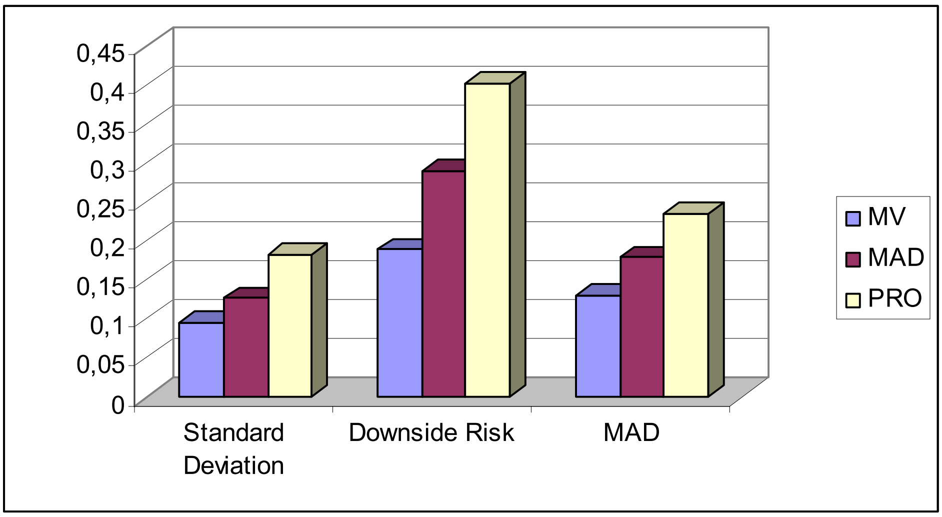
Even though an ordinary investor generally evaluates his/her investment performance only according to the returns of the portfolio, for a valid evaluation, calculating the amount of the return for every unit of risk borne would be more appropriate. To calculate the reward per risk equation 13 is employed:

Portfolio Performance = mean return/risk (13)

It is obvious that portfolio performance calculated using equation (13) will differ according to the risk measure chosen. In this paper 3 different risk measures are used. These are standard deviation, downside risk and mean absolute deviation. For every portfolio and for every investment horizon, 3 performance measures are calculated using these 3 different risk measures. Besides, well known portfolio performance measure, Sharpe ratio is calculated.

The performance of the portfolios which were modified every month is shown in Figure 2. Compared with the mean returns in Figure 1, the difference between the performances of different portfolios is smaller. This indicates that, independent of the risk measure selected, portfolios composed according to the proposed model are riskier than the other portfolios.

Figure 2: Different Risk Measures of the Portfolios



Using the proposed model, the increased risk for portfolios is seen not only in those portfolios revised monthly but also in all investment horizons. Table 1 summarizes some statistics of the portfolios. Another conclusion of the table is that portfolios using the proposed model outperform the other portfolios for investment horizons shorter than 1 year.

Table 1: Summary Statistics of the Portfolios

	Optimization Model	Mean Return	Standard Deviation	Downside Risk	MAD
1 Month	MV	0,012	0,126	0,064	0,093
	MAD	0,015	0,119	0,052	0,084
	PRO	0,033	0,182	0,082	0,141
3 Months	MV	0,016	0,132	0,067	0,099
	MAD	0,012	0,132	0,073	0,092
	PRO	0,018	0,178	0,082	0,132
6 Months	MV	0,013	0,127	0,067	0,096
	MAD	0,016	0,141	0,075	0,103
	PRO	0,027	0,173	0,079	0,133

When compared to mean variance and mean absolute deviation models, the new proposed model generates more risky portfolios. But for individual investors who generally make investment decisions based on return proposed model constitutes more attractive portfolios. Table 2 summarizes the performance measures of the portfolios. For 1 and 6 month periods, proposed model outperforms the other portfolios. For 3 month period, mean variance portfolio shows a better performance although proposed model's mean return is the highest.

Table 2: Performance Measures of the Portfolios

	Optimization Model	Sharpe Ratio	Mean Return / Downside Risk	Mean Return / MAD
1 Month	MV	0,096	0,188	0,129
	MAD	0,126	0,288	0,179
	PRO	0,182	0,402	0,234
3 Months	MV	0,118	0,239	0,162
	MAD	0,091	0,164	0,130
	PRO	0,104	0,220	0,136
6 Months	MV	0,100	0,194	0,135
	MAD	0,116	0,213	0,155
	PRO	0,159	0,342	0,203

When compared to mean variance and mean absolute deviation models, the new proposed model generates more risky portfolios. But for individual investors who generally make investment decisions based on return proposed model constitutes more attractive portfolios.

IV. CONCLUDING REMARKS

Most of the research about portfolio optimization after Markowitz focused on overcoming the computational burden of the mean variance model. Konno and Yamazaki suggest a new model which accepts MAD as a risk measure instead of variance. They also proved that minimizing MAD is similar to minimizing variance if the returns of the stocks are multivariate normally distributed.

Markowitz's mean variance model and the mean absolute deviation models accept risk as the deviation that can be positive (upside) or negative (downside) from the expected return. In other words both of the models punish not only the negative (downside) deviations but also the positive (upside) deviations. However, the difference between negative and positive deviations is crucial.

Negative deviation is accepted to be undesirable for most of the investor while positive deviation is desirable.

In this paper a new model that takes into consideration both risk downside deviations and better investment opportunities upside deviations-, is proposed. The difference between the proposed model and the other portfolio optimization models is their objectives. The proposed model assumes that an investor wants to choose a portfolio with higher upside deviations and lower downside deviations.

In this paper this theoretically right issue is tested with real data of an emerging market. The results of the analysis showed that the portfolios of the proposed model are riskier but generate higher returns. So this model can be useful for investors whose investment evaluation mostly depends on return.

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